7.6 Surface integrals of vector fields

We learn

- an integral of a vector field F over a parametrized surface
- Interpretation of this integral as flux across the surface
- What an orientation of a surface is
- Some surfaces cannot be oriented
- How a parametrization determines an orientation
- Practice evaluating these integrals

What we can do without:

- Most of the formulas at the end of 7.6, on page 411.
- It is not worth remembering special formulas for surfaces that are graphs, or for spheres.

The definition

We take

- a vector field F : R^3 -> R^3
- a parametrized surface
  Phi : D -> R^3 with Phi (D) = S

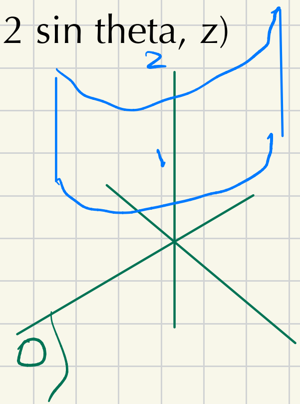
We define what the book calls the surface integral of F over Phi. I would prefer to call it the integral of the flux form of F, or the flux integral of F.

$$\iint_{\Phi} F \cdot d\underline{S} = \iint_{D} F \cdot (T_u \times T_v)\, du\, dv$$

Later, when we know what an orientation of S is, we might write:

$$\iint_{S} F \cdot d\underline{S}$$

Example:
Find the flux of the vector field
F(x,y,z) = (y z, x^2, x^2yz) across the half-cylinder
Phi(theta, z) = (2 cos theta, 2 sin theta, z)
0 ≤ theta ≤ π, 1 ≤ z ≤ 2



$$T_\theta = (-2\sin\theta, 2\cos\theta, 0)$$

$$T_z = (0, 0, 1)$$

$$T_\theta \times T_z = (2\cos\theta, 2\sin\theta, 0)$$

The flux is

$$\int_1^2 \int_0^\pi (2\sin\theta\, z, 4\cos^2\theta, 4\cos^2\theta\, 2\sin\theta\, z) \cdot (2\cos\theta, 2\sin\theta, 0)\, d\theta\, dz$$

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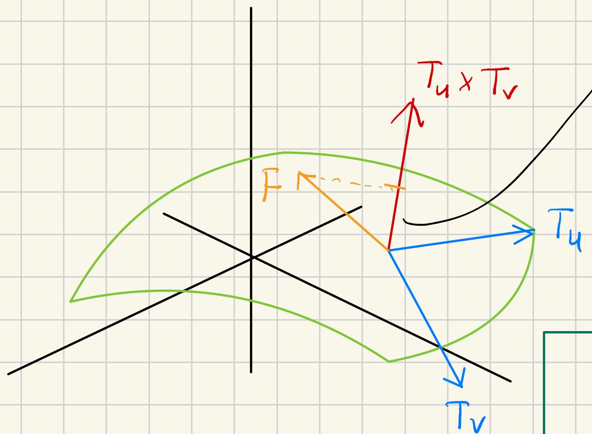
What does it mean?



Length

$$\frac{F\cdot(T_u\times T_v)}{\|T_u\times T_v\|}$$

$$\iint_D F\cdot(T_u\times T_v)\,du\,dv$$

$$= \iint_D \frac{F\cdot(T_u\times T_v)}{\|T_u\times T_v\|}\,\|T_u\times T_v\|\,du\,dv$$

$$= \iint_D F\cdot \underline{n}\,\|T_u\times T_v\|\,du\,dv = \int_S F\cdot\underline{n}\,dS$$

Let $\underline{n} = \dfrac{T_u\times T_v}{\|T_u\times T_v\|}$

This is a unit normal vector for S

Pre-class Warm-up!!!

The picture on the left appeared at the end of class on Friday. Was it intended to help us understand

a. how to set up an integral

b. how to find the area of a surface

c. what the flux of a vector field is

d. why the the integral computes the flux of the vector field across the surface

e. none of the above.

The second half of section 7.6: Orientations

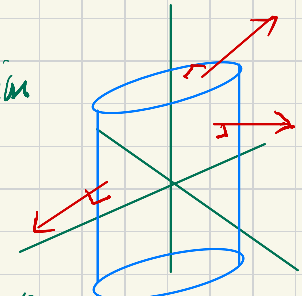
We learn:

- what is an orientation?
- Descriptions like “the normal points out”.
- A parametrization determines an orientation
- Terminology: consistent or compatible with the orientation
- Unit normal
- We can only do flux integrals on orientable surfaces.
- Theoretical things, not proved: the integral does not depend on the choice of parametrization, provide it is consistent for the orientation.
- We don’t need theorem 5 or Gauss’s law of the special formulas that arise when the surface is a graph.
- Pages 410 and 411 we only need 1a and 1b.

Orientation of surfaces

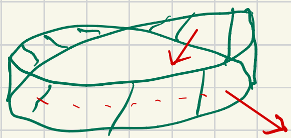
Definition: An orientation of a surface S is a continuous choice of a normal vector at each point of S.

A cylinder has an orientation



This orientation points out.
The orientation points in.

Some surfaces do not have an orientation.
The Möbius strip has no orientation!



If S has an orientation we say it is orientable and then S has precisely two orientations (if it is connected)

Parametrizations Phi : D -> R^3 determine orientations.

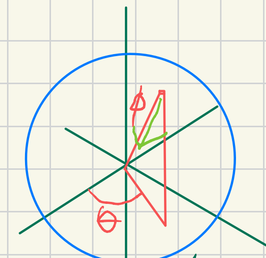
Sphere: $\Phi(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$T_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$

$T_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$

$T_\theta \times T_\phi = (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi)$

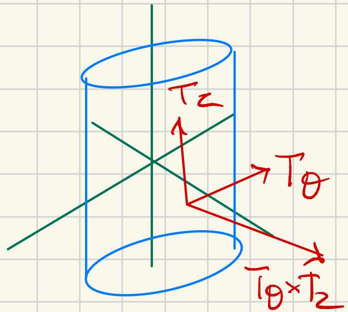
This normal vector points in to the sphere (because coords are < 0 when θ, ϕ are small)



More usually we take an outward pointing orientation.
The parametrization $\Phi_1(\phi,\theta) =$ (same as at top) gives an outward orientation

Cylinder

$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$



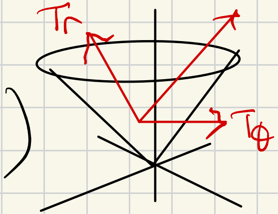
$T_\theta = (-\sin\theta, \cos\theta, 0)$

$T_z = (0, 0, 1)$

$T_\theta \times T_z = (\cos\theta, \sin\theta, 0)$

points out.

Cone



$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r)$

$0 \le \theta \le 2\pi, \quad 0 \le r$

Does the orientation determined by Phi point up or down?

a. Up

b. Down

c. Neither

Example.

Find $\iint_S F \cdot dS$ where $F(x,y,z) = (0, 0, z)$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ where $z \leq 1$, oriented by a downward pointing normal.

Same picture as last page.

Solution:

Issue: The parametrization $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$ is wrong for the downward orientation.

2 approaches

1. Do the calculation with Φ and multiply by -1 at the end.
2. Get a parametrization with the correct orientation

$$\Phi_1(\theta, r) = (r\cos\theta, r\sin\theta, r).$$

Right-hand rule: